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Weak Korovkin Approximation by Completely Positive Linear Maps on $\beta(H)$

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A subset $\mathscr{S}$ of $\beta(H)$ is called a *weak Korovkin set* if, for every net (ϕ_α) of completely positive linear maps on $\beta(H)$ with $\phi_\alpha(I) \leqslant I$, the relation $\phi_\alpha(S) \to S$ weakly, $S \in \mathscr{S}$, implies $\phi_\alpha(T) \to T$ weakly, $T \in \beta(H)$. Let $\mathscr{S}$ be an irreducible subset of $\beta(H)$ which contains the identity operator.

THEOREM. (i) *If there are* $T \in \operatorname{span}\{\mathscr{S} + \mathscr{S}^*\}$ *and a compact operator* K *such that* $\|T-K\| < \|T\|$, *then* $\mathscr{S}$ *is a weak Korovkin set.* (ii) *If* $C^*(\mathscr{S})$ *contains a nonzero compact operator, then* $\mathscr{E} = \{S: S \in \mathscr{S}\} \cup \{S^*S + SS^*: S \in \mathscr{S}\}$ *is a weak Korovkin set. In particular, if* $\mathscr{S} = \{I, S_1, \ldots, S_m\}$, *then* $\mathscr{E}' = \{I, S_1, \ldots, S_m, \sum_{j=1}^m (S_j^* S_j + S_j S_j^*)\}$ *is a weak Korovkin set.* © 1984 Academic Press, Inc.

1. INTRODUCTION

Let X be a compact Hausdorff space, and denote by $C(X)$ the set of all complex-valued continuous functions on X. A subset $\mathscr{S}$ of $C(X)$ is called a *Korovkin set* in $C(X)$ if, for each sequence (ϕ_n) of positive linear maps on $C(X)$, the relation $\phi_n(g) \to g$ uniformly, $g \in \mathscr{S}$, implies $\phi_n(f) \to f$ uniformly, $f \in C(X)$. A famous result of Korovkin states [7, p. 16] that for $X = [a, b]$, the set $\{1, x, x^2\}$ is a Korovkin set. This result has been extended as follows. For $x \in X$, consider the evaluation functional $\varepsilon_x(f) = f(x)$, $f \in C(X)$. Let $\mathscr{S} \subset C(X)$ contain the constant function 1 and separate the points of X. Then $\mathscr{S}$ is a Korovkin set in $C(X)$ if and only if for every $x \in X$, the restriction $\varepsilon_x|_{\mathscr{S}}$ has a unique positive linear extension to $C(X)$ [4, Corollary 1, p. 167]. In particular, if the functions $g_1, \ldots, g_m$ in $C(X)$ separate the points of X, then $\{1, g_1, \ldots, g_m, \sum_{j=1}^m |g|^2\}$ is a Korovkin set in $C(X)$ [4, Example 2, p. 180; 9, Theorem 2.8].

In [11], the following analogue of Korovkin's result for a C^*-algebra A with identity 1_A is proved. Let $1_A \in \mathscr{S} \subset A$. If for every ε in the pure state space of A, the restriction $\varepsilon|_{\mathscr{S}}$ has a unique positive linear extension to A, then $\mathscr{S}$ is a Korovkin set in A. The applicability of this analogue is,

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however, severely restricted by the lack of exact knowledge of the pure state space of a noncommutative C^*-algebra.

The purpose of the present paper is to obtain analogues of Korovkin's result for the set $\beta(H)$ of all bounded operators on a complex Hilbert space H, when approximation in the weak sense by completely positive linear maps is considered. We introduce the notion of a weak Korovkin set in $\beta(H)$. For an irreducible set $\mathscr{S}$ in $\beta(H)$ to be a weak Korovkin set, it turns out to be crucial that the identity representation of $C^*(\mathscr{S})$ be a boundary representation for $\mathscr{S}$ in the sense of Arveson (Theorem 3.2). We give several sufficient conditions for this to happen (Corollaries 3.4 and 3.7). Particular cases of these results are cited which involve some well-known bounded operators on $L^2([0, 1])$ and on l^2 (Examples 3.5 and 3.8). Here, convergence on a set of two or three bounded operators implies convergence on all bounded operators. This work improves upon some results proved by the authors in Example 5(iii) of [8] as far as the weak convergence is concerned. In [8] convergence on only the compact operators was obtained, and that too under more restrictive assumptions.

2. Completely Positive Maps on $\beta(H)$

In this section, we recall some notions from the theory of C^*-algebras and prove a unique extension result for certain maps on the C^*-algebra $\beta(H)$ of all bounded operators on a Hilbert space H.

Let A and B denote C^*-algebras with identities 1_A and 1_B, respectively. A linear map $\phi: A \to B$ is called *completely positive* if for every natural number k, $a_1,..., a_k$ in A and $b_1,..., b_k$ in B, we have

$$\sum_{i,j=1}^{k} b_i^* \phi(a_i^* a_j)\, b_j = b^* b$$

for some $b \in B$. The set of all such maps is denoted by $CP(A, B)$ which is abbreviated to $CP(A)$ in case $B = A$. By considering $k = 1$ and $b_1 = 1_B$, we see that every completely positive linear map is positive; the converse is true if either A or B is commutative (3.4, 3.5, and 3.9 of Chap. IV in [12]). Thus, completely positive linear maps on $\beta(H)$ are appropriate analogues of positive linear maps on $C(X)$.

Stinespring has proved [1, p. 145] that $\phi \in CP(A, \beta(H))$ if and only if there is a complex Hilbert space G, a representation (i.e., a *homomorphism) $\pi: A \to \beta(G)$ and a bounded linear map $V: H \to G$ such that

$$\phi(a) = V^* \pi(a)\, V, \qquad a \in A$$

and

$$[\pi(A)\, V(H)] = G,$$

where $[F]$ denotes the closed subspace generated by $F \subset G$. Further, the above G, π, and V are uniquely determined by ϕ up to unitary equivalence.

A map $\phi \in CP(A, B)$ is said to be *pure* if the relation $\phi = \phi_1 + \phi_2$, $\phi_1, \phi_2 \in CP(A, B)$ implies that ϕ_1 and ϕ_2 are scalar multiples of ϕ. A subset $\mathscr{S}$ of $\beta(H)$ is called *irreducible* if the only closed subspaces of H left invariant by all S and S^*, $S \in \mathscr{S}$, are $\{0\}$ and H. A nonzero representation $\pi: A \to \beta(H)$ is said to be irreducible if its range $\pi(A)$ is an irreducible subset of $\beta(H)$ [3, p. 14]. Arveson has proved [1, p. 161] that if $0 \neq \phi \in CP(A, \beta(H))$ is pure, then π is ireducible and $V \neq 0$, in the Stinespring representation of ϕ; conversely, if $\pi: A \to \beta(G)$ is an irreducible representation and V is a nonzero bounded linear map from H to G, then $\phi(\cdot) = V^*\pi(\cdot)V$ is a pure map.

Let $\kappa(H)$ denote the set of all compact operators on H. Since $\kappa(H)$ is weakly dense in $\beta(H)$, it is clear that if $0 \neq \phi \in CP(\beta(H))$ is continuous in the weak topology on $\beta(H)$, then ϕ is not zero on $\kappa(H)$. Although the converse is false in general, it holds for a pure map.

LEMMA 2.1. *Let $0 \neq \phi_0 \in CP(\beta(H))$ be pure. If ϕ_0 is not zero on $\kappa(H)$, then ϕ_0 is continuous in the weak topology on $\beta(H)$.*

Proof. Consider the Stinespring representation

$$\phi_0(T) = V^*\pi(T)V, \qquad T \in \beta(H),$$

of ϕ_0. Since ϕ_0 is pure, the representation $\pi: \beta(H) \to \beta(G)$ is irreducible. Now, $\pi(\kappa(H)) \neq \{0\}$ and $\kappa(H)$ is an ideal in $\beta(H)$. Hence by Theorem 1.3.4 on [3], the restriction $\pi|_{\kappa(H)}$ is irreducible and extends uniquely to an irreducible representation of $\beta(H)$. But every irreducible representation of $\kappa(H)$ is unitarily equivalent to the identity representation id: $\kappa(H) \to \beta(H)$ [3, p. 20]. Thus, there is a unitary map $U: G \to U$ such that

$$\pi(T) = U^*TU, \qquad T \in \beta(H).$$

Hence

$$\phi_0(T) = (UV)^*T(UV), \qquad T \in \beta(H).$$

If T_α, $T \in \beta(H)$ and $\langle T_\alpha x, y\rangle \to \langle Tx, y\rangle$ for all $x, y \in H$, then

$$\begin{aligned}\langle \phi_0(T)x, y\rangle &= \langle T_\alpha UVx, UVy\rangle \\ &\to \langle TUVx, UVy\rangle \\ &= \langle \phi_0(T)x, y\rangle.\end{aligned}$$

Thus, ϕ_0 is continuous in the weak topology on $\beta(H)$. ∎

PROPOSITION 2.2. *Let $\mathscr{C}$ be an irreducible C^*-subalgebra of $\beta(H)$ containing the identity operator I and a nonzero compact operator. Consider the identity map* $\mathrm{id}: \beta(H) \to \beta(H)$. *Then* $\mathrm{id}|_{\mathscr{C}}$ *has a unique completely positive linear extension to* $\beta(H)$, *namely,* id *itself.*

Proof. Let $\phi \in CP(\beta(H)$ and $\phi(T) = \mathrm{id}(T) = T$ for all $T \in \mathscr{C}$. If $\phi \neq \mathrm{id}$, then the set

$$Q = \{\psi \in CP(\beta(H)) \colon \psi|_{\mathscr{C}} = \mathrm{id}|_{\mathscr{C}}\}$$

contains at least two elements. This set is convex and since $\|\psi\| = \|\psi(I)\| = 1$ for every $\psi \in Q$, it is compact in the weak operator topology [6, p. 974]. By the Krein–Milman theorem, Q must contain an extreme point $\phi_0 \neq \mathrm{id}$.

We prove that ϕ_0 is, in fact, pure. Let $\phi_0 = \phi_1 + \phi_2$ with $0 \neq \phi_i \in CP(\beta(H))$. Then $\phi_0|_{\mathscr{C}} = \phi_1|_{\mathscr{C}} + \phi_2|_{\mathscr{C}}$, where $\phi_i|_{\mathscr{C}} \in CP(\mathscr{C}, \beta(H))$, and $\phi_i|_{\mathscr{C}} \neq 0$, since $\mathscr{C}$ contains I. Now,

$$\phi_0|_{\mathscr{C}}(\cdot) = \mathrm{id}|_{\mathscr{C}}(\cdot) = I^*\pi_0(\cdot)I,$$

where $\pi_0 \colon \mathscr{C} \to \beta(H)$ is given by $\pi_0(T) = T$. Since $\mathscr{C}$ is an irreducible C^*-subalgebra of $\beta(H)$, it follows that π_0 is irreducible so that $\phi_0|_{\mathscr{C}}$ is pure. Hence

$$\phi_1|_{\mathscr{C}} = \alpha_1 \phi_0|_{\mathscr{C}} \qquad \text{and} \qquad \phi_2|_{\mathscr{C}} = \alpha_2 \phi_0|_{\mathscr{C}}$$

for some nonnegative scalars α_1 and α_2. Since $\phi_i|_{\mathscr{C}} \neq 0$, we see that $\alpha_i > 0$. Also, $\alpha_1 + \alpha_2 = 1$, since

$$0 \neq \phi_0(I) = \phi_1(I) + \phi_2(I) = (\alpha_1 + \alpha_2)\,\phi_0(I).$$

Now, ϕ_i/α_i belongs to Q and

$$\phi_0 = \alpha_1(\phi_1/\alpha_1) + \alpha_2(\phi_2/\alpha_2).$$

As ϕ_0 is an extreme point of Q, it follows that $\phi_1/\alpha_1 = \phi_0 = \phi_2/\alpha_2$, i.e., ϕ_0 is pure.

Again, $\phi_0(K) = \mathrm{id}(K) = K \neq 0$ for some compact $K \in \mathscr{C}$. By Lemma 2.1, ϕ_0 is continuous in the weak topology on $\beta(H)$. Since $\mathscr{C}$ is an irreducible C^*-algebra and contains a nonzero compact operator, it contains $\kappa(H)$ [3, p. 18]. Thus, the weakly continuous maps ϕ_0 and id agree on $\kappa(H)$, which is weakly dense in $\beta(H)$. This shows that $\phi_0 = \mathrm{id}$, and the proof is complete. ∎

The proofs of Lemma 2.1 and Proposition 2.2 should be compared with the proof of Theorem 2.4.5 of [1, p. 180].

3. Weak Korovkin Sets in $\beta(H)$

In this section we consider irreducible subsets of $\beta(H)$ and find a variety of conditions under which weak convergence of a net of completely positive linear maps on such a subset implies its weak convergence on the entire C^*-algebra $\beta(H)$.

Definition 3.1. A subset $\mathscr{S}$ of $\beta(H)$ will be called *a weak Korovkin set in* $\beta(H)$ if, for each net (ϕ_α) in $CP(\beta(H))$ satisfying $\phi_\alpha(I) \leqslant I$, the relation $\phi_\alpha(S) \to S$ weakly, $S \in \mathscr{S}$, implies $\phi_\alpha(T) \to T$ weakly, $T \in \beta(H)$.

We remark that if $\mathscr{S}$ is a weak Korovkin set in $\beta(H)$, then in fact $\phi_\alpha(T) \to T$ *strongly* for every $T \in \beta(H)$. This follows by noting that for $\phi \in CP(\beta(H))$ with $\phi(I) \leqslant I$, we have

$$\begin{aligned}\phi(T)^*\phi(T) &\leqslant \|\phi\|\,\phi(T^*T)\\ &= \|\phi(I)\|\,\phi(T^*T)\\ &\leqslant \phi(T^*T), \qquad T \in \beta(H),\end{aligned}$$

by Corollary 3.8, p. 199 of [12]. Hence for $x \in H$,

$$\begin{aligned}\|\phi_\alpha(T)x - Tx\|^2 &= \langle \phi_\alpha(T^*)\,\phi_\alpha(T)x, x\rangle + \langle T^*Tx, x\rangle\\ &\quad - 2\,\mathrm{Re}\langle \phi_\alpha(T)x, Tx\rangle\\ &\leqslant \langle \phi_\alpha(T^*T)x, x\rangle + \langle T^*Tx, x\rangle\\ &\quad - 2\,\mathrm{Re}\langle \phi_\alpha(T)x, Tx\rangle\\ &\to \langle T^*Tx, x\rangle + \langle T^*Tx, x\rangle - 2\,\mathrm{Re}\langle Tx, Tx\rangle\\ &= 0.\end{aligned}$$

For $\mathscr{S} \subset \beta(H)$, we denote the C^*-subalgebra generated by I and $\mathscr{S}$ in $\beta(H)$ by $C^*(\mathscr{S})$.

Theorem 3.2. *Let $\mathscr{S}$ be an irreducible set in $\beta(H)$ such that $\mathscr{S}$ contains the identity operator I and $C^*(\mathscr{S})$ contains a nonzero compact operator. Then $\mathscr{S}$ is a weak Korovkin set in $\beta(H)$ if and only if* $\mathrm{id}|_{\mathscr{S}}$ *has a unique completely positive linear extension to $C^*(\mathscr{S})$, namely,* $\mathrm{id}|_{C^*(\mathscr{S})}$.

Proof. Let $\mathscr{S}$ be a weak Korovkin set in $\beta(H)$. Let $\phi \in CP(C^*(\mathscr{S}), \beta(H))$ and $\phi|_{\mathscr{S}} = \mathrm{id}|_{\mathscr{S}}$. Then ϕ can be extended to $\tilde{\phi} \in CP(\beta(H))$ [1, Theorem 1.2.3]. By considering the constant net $\phi_\alpha = \tilde{\phi}$ in $CP(\beta(H))$ with $\phi_\alpha(I) = \tilde{\phi}(I) = \mathrm{id}(I) = I$, we see that $\phi(T) = \tilde{\phi}(T) = \phi_\alpha(T) \to T$ for every $T \in C^*(\mathscr{S})$. Thus, $\phi = \mathrm{id}|_{C^*(\mathscr{S})}$. This proves the necessity half.

To improve the sufficiency half, assume that $\mathrm{id}|_{\mathscr{S}}$ has a unique completely positive linear extension to $C^*(\mathscr{S})$. But by Proposition 2.2, $\mathrm{id}|_{C^*(\mathscr{S})}$ has a unique completely positive linear extension to $\beta(H)$. Thus, $\mathrm{id}|_{\mathscr{S}}$ has a unique completely positive linear extension to $\beta(H)$, namely, id itself. Now let (ϕ_α) be a net in $CP(\beta(H))$ satisfying $\phi_\alpha(I) \leqslant I$, and let $\phi_\alpha(S) \to S$ weakly for every $S \in \mathscr{S}$. On the set $\{\psi \in CP(\beta(H)): \psi(I) \leqslant I\}$, consider the weak operator topology, in which the set is compact. If ϕ is any cluster point of the net (ϕ_α), let (ϕ_β) be a subnet of (ϕ_α) converging to ϕ. Then for every $S \in \mathscr{S}$,

$$\phi(S) = \lim \phi_\beta(S) = \lim \phi_\alpha(S) = S,$$

i.e., $\phi|_{\mathscr{S}} = \mathrm{id}|_{\mathscr{S}}$. Hence $\phi = \mathrm{id}$ on $\beta(H)$. Thus, every cluster point of the net (ϕ_α) coincides with id, i.e., $\phi_\alpha \to \mathrm{id}$ in the weak operator topology, or $\phi_\alpha(T) \to T$ weakly for every $T \in \beta(H)$. ∎

Remark 3.3. The condition "$\mathrm{id}|_{\mathscr{S}}$ has a unique completely positive linear extension to $C^*(\mathscr{S})$" is sometimes expressed by saying that the identity representation of $C^*(\mathscr{S})$ is a *boundary representation for* $\mathscr{S}$. The famous boundary theorem of Arveson (Theorem 2.1.1 of [2]) states as follows: Let $\mathscr{S}$ be an irreducible set in $\beta(H)$ such that $\mathscr{S}$ contains the identity operator and $C^*(\mathscr{S})$ contains a nonzero compact operator. Then the identity representation of $C^*(\mathscr{S})$ is a boundary representation for $\mathscr{S}$ if and only if the quotient map $q: \beta(H) \to \beta(H)/k(H)$ is not completely isometric on span $\{\mathscr{S} + \mathscr{S}^*\}$, i.e., for some natural number k, if M_k denotes the set of all $k \times k$ complex matrices, then the map $q \otimes \mathrm{id}_k: C^*(\mathscr{S}) \otimes M_k \to ((C^*(\mathscr{S})/k(H)) \otimes M_k$ is not isometric. This result (especially, the case $k = 1$) provides a useful method by which a set $\mathscr{S}$ in $\beta(H)$ can be shown to be a weak Korovkin set.

COROLLARY 3.4. *Let $\mathscr{S}$ be an irreducible set in $\beta(H)$ which contains the identity operator I. Suppose that there are $T \in \mathrm{span}\{\mathscr{S} + \mathscr{S}^*\}$ and a compact operator K in $\beta(H)$ such that $\|T - K\| < \|T\|$. Then $\mathscr{S}$ is a weak Korovkin set in $\beta(H)$.*

Proof. As noted in the proof of the Corollary on p. 289 of [2], the irreducible C^*-algebra $C^*(\mathscr{S})$ must contain a nonzero compact operator, for otherwise the quotient map $q: \beta(H) \to \beta(H)/k(H)$ would be injective. But a * isomorphism between two C^*-algebras is an isometry (Propositions 5.2 and 5.3, pp. 21–22 of [12]). Hence q is isometric on $\mathrm{span}\{\mathscr{S} + \mathscr{S}^*\}$. This would contradict $\|T - K\| < \|T\|$. By the boundary theorem of Arveson quoted in Remark 3.3, we see that $\mathrm{id}|_{\mathscr{S}}$ has a unique completely positive linear extension to $C^*(\mathscr{S})$. Hence Theorem 3.2 shows that $\mathscr{S}$ is a weak Korovkin set in $\beta(H)$. ∎

EXAMPLES 3.5. If $\mathscr{S}$ is an irreducible set of operators in $\beta(H)$ containing I, and if span$\{\mathscr{S}+\mathscr{S}^*\}$ (or, $\mathscr{S}$ itself) contains a compact operator, then the requirement of Corollary 3.4 is trivially satisfied, and we see that $\mathscr{S}$ is a weak Korovkin set.

For example, let S be a compact irreducible operator on $\beta(H)$. Then $\mathscr{S}=\{I, S\}$ is a weak Korovkin set in $\beta(H)$. The Volterra integration operator $V(f)(u)=\int_0^u f(t)\,dt$, $f\in L^2([0,1])$, $u\in[0,1]$ is a compact irreducible operator on $H=L^2([0,1])$. Thus, weak convergence of a net in $CP(\beta(H))$ on only two operators I and V implies its weak convergence on every bounded operator on $L^2([0,1])$.

For f in $L^2([0,1])$, and $u\in[0,1]$, let

$$M(f)(u)=uf(u),$$

$$T(f)(u)=\left(\int_0^1 f(t)\,dt\right)u.$$

Then $\mathscr{S}=\{I, M, T\}$ is an irreducible set (p. 245 of [8]) and T is compact. Hence convergence on I, M, and T implies convergence on $\beta(L^2([0,1])$.

These results improve upon the results given in Example 5(iii) of [8], where by assuming convergence on $\{I, V, V^*V+VV^*\}$ or $\{I, M, T, 2M^2+T^*T+TT^*\}$, convergence only on compact operators on $L^2([0,1])$ was obtained.

Before we derive another interesting corollary from Theorem 3.2, we prove another unique extension result which may be of independent interest. Let A and B be C^*-algebras. A * linear map $\phi: A\to B$ is called a *Schwarz map* if

$$\phi(a)^*\phi(a)\leqslant\phi(a^*a),\qquad a\in A.$$

PROPOSITION 3.6. *Let E be a subset of a C^*-algebra A, and let C denote the C^*-subalgebra generated by E in A. Let $F=\{a: a\in E\}\cup\{a^*a+aa^*: a\in E\}$. Consider a *homomorphism ϕ from C to a C^*-algebra B. Then $\phi|_F$ has a unique Schwarz extension to C, namely, ϕ itself.*

Proof. Let $\psi: C\to B$ be a Schwarz map such that $\psi|_F=\phi|_F$. Let

$$C_\psi=\{a\in A: \psi(a)=\phi(a), \psi(a^*a+aa^*)=\phi(a^*a+aa^*)\}.$$

For $a\in C_\psi$,

$$\phi(a^*a)=\phi(a)^*\phi(a)=\psi(a)^*\psi(a)\leqslant\psi(a^*a),$$

and similarly, $\phi(aa^*)\leqslant\psi(aa^*)$. But

$$[\psi(a^*a)-\phi(a^*a)]+[\psi(aa^*)-\phi(aa^*)]=\psi(a^*a+aa^*)-\phi(a^*a+aa^*)$$
$$=0.$$

Hence $\psi(a^*a) = \phi(a^*a)$ and $\psi(aa^*) = \phi(aa^*)$. Thus, we have

$$C_\psi = \{a \in A : \psi(a) = \phi(a), \psi(a^*a) = \phi(a^*a), \psi(aa^*) = \phi(aa^*)\}.$$

Next, we show that C_ψ is a C^*-subalgebra of A. Clearly C_ψ is closed under $*$ and scalar multiplication. We prove that if $a \in C_\psi$ and $b \in A$ with $\psi(b) = \phi(b)$, then $\psi(ab) = \phi(ab)$ and $\psi(ba) = \phi(ba)$. Now, for every positive number t,

$$\begin{aligned} t[\psi(b)\,\psi(a) + \psi(a)^*\psi(b)^*] &= \psi((tb^* + a))\,\psi(tb^* + a) \\ &\quad - t^2\psi(b)\,\psi(b^*) - \psi(a^*)\,\psi(a) \\ &\leqslant \psi((tb^* + a)^*(tb^* + a)) \\ &\quad - t^2\psi(b)\,\psi(b^*) - \psi(a^*)\,\psi(a) \\ &= t\psi(ba + a^*b^*) \\ &\quad + t^2[\psi(bb^*) - \psi(b)\,\psi(b^*)], \end{aligned}$$

since $a \in C_\psi$. Hence

$$\psi(b)\,\psi(a) + \psi(a)^*\psi(b)^* - \psi(ba + a^*b^*) \leqslant t[\psi(bb^*) - \psi(b)\,\psi(b^*)].$$

Since this is true for every $t > 0$, we have

$$\psi(b)\,\psi(a) + \psi(a)^*\psi(b)^* - \psi(ba + a^*b^*) \leqslant 0.$$

Changing a to $-a$, we have

$$-\psi(b)\,\psi(a) - \psi(a)^*\psi(b)^* + \psi(ba + a^*b^*) \leqslant 0.$$

Hence

$$\psi(b)\,\psi(a) + \psi(a)^*\psi(b)^* = \psi(ba + a^*b^*).$$

Changing a to ia, we have

$$\psi(b)\,\psi(a) - \psi(a)^*\psi(b)^* = \psi(ba - a^*b^*).$$

Thus, by adding we obtain

$$\psi(ba) = \psi(b)\,\psi(a) = \phi(b)\,\phi(a) = \phi(ba).$$

Changing a to a^* and b to b^*, we have

$$\psi(b^*a^*) = \phi(b^*a^*),$$

and taking adjoints, we obtain

$$\psi(ab) = \phi(ab).$$

By repeated application of this result, it is easy to see that for $a, b \in C_\psi$, we have $a+b$, $ab \in C_\psi$. Also, C_ψ is closed in A since ψ and ϕ are continuous. Thus, C_ψ is a C^*-subalgebra of A. But $E \subset C_\psi$ by the definition of F and the assumption that $\psi|_F = \phi|_F$. Hence the C^*-algebra C generated by E in A is contained in C_ψ so that $\psi(a) = \phi(a)$ for every $a \in C$. ∎

The proof of the above proposition closely follows the one given in [10]. We have written it out in detail because it is much simpler in the present case. If $B = \beta(H)$, the above result can also be stated as follows: ϕ is a boundary representation for F.

Corollary 3.7. *Let $\mathscr{S}$ be an irreducible set in $\beta(H)$ such that $\mathscr{S}$ contains the identity operator I and $C^*(\mathscr{S})$ contains a nonzero compact operator. Then*

$$\mathscr{E} = \{S : S \in \mathscr{S}\} \cup \{S^*S + SS^* : S \in \mathscr{S}\}$$

is a weak Korovkin set in $\beta(H)$.

If $\mathscr{S}$ is finite and $\mathscr{S} = \{I, S_1, ..., S_m\}$, then

$$\mathscr{E}' = \left\{I, S_1, ..., S_m, \sum_{j=1}^{m} (S_j^*S_j + S_jS_j^*)\right\}$$

is a weak Korovkin set in $\beta(H)$.

Proof. By Proposition 3.6 with $A = B = \beta(H)$, $E = \mathscr{S}$, and $\phi = \mathrm{id}|_{C^*(\mathscr{S})}$, we see that $\mathrm{id}|_{\mathscr{E}}$ has a unique Schwarz extension to $C^*(\mathscr{S}) = C^*(\mathscr{E})$. Let $\psi \in CP(C^*(\mathscr{S}), \beta(H))$, and $\psi|_{\mathscr{E}} = \mathrm{id}|_{\mathscr{E}}$. Then ψ is a Schwarz map by Corollary 3.8 on p. 199 of [12]. Thus, $\mathrm{id}|_{\mathscr{E}}$ has a unique completely positive linear extension to $C^*(\mathscr{E})$. The desired result now follows by Theorem 3.2.

If $\mathscr{S} = \{I, S_1, ..., S_m\}$, and ψ is a Schwarz map on $C^*(\mathscr{S})$ with $\psi|_{\mathscr{E}'} = \mathrm{id}|_{\mathscr{E}'}$, then it can be easily seen that $\psi|_{\mathscr{E}} = \mathrm{id}|_{\mathscr{E}}$, where $\mathscr{E} = \{I, S_1, ..., S_m, S_1^*S_1 + S_1S_1^*, ..., S_m^*S_m + S_mS_m^*\}$. This implies that $\mathscr{E}'$ is a weak Korovkin set in $\beta(H)$. ∎

Examples 3.8. Let S be an irreducible operator which is almost normal (i.e., $S^*S - SS^*$ is a compact operator), but not normal (i.e., $S^*S - SS^* \neq 0$). Then the set $\mathscr{S} = \{I, S\}$ satisfies the requirement of Corollary 3.7 and we see that the set

$$\mathscr{E} = \{I, S, S^*S + SS^*\}$$

of three operators is a weak Korovkin set.

We illustrate this result by describing a class of irreducible almost normal but nonnormal operators. Let H be a separable Hilbert space and let

$\{e_0, e_1, ...\}$ be an orthonormal basis for H. A *unilateral weighted shift operator* S on H is defined by $Se_n = \alpha_n e_{n+1}$, $n = 0, 1, ...$, where $0 < |\alpha_n| \leqslant \alpha < \infty$. Each such operator is irreducible. Also, $(S^*S - SS^*) e_n = (|\alpha_n|^2 - |\alpha_{n-1}|^2) e_n$ for $n = 0, 1, 2, ...$, with $\alpha_1 = 0$. Thus, S is almost normal iff $|\alpha_n| - |\alpha_{n-1}| \to 0$ as $n \to \infty$, and S is always nonnormal since $(S^*S - SS^*) e_0 = |\alpha_0|^2 e_0 \neq 0$. If $\alpha_n \to 0$, then S is itself compact (and irreducible).

For a unilateral weighted shift S with weights α_n, it can be easily seen that $\|S\| = \sup_n |\alpha_n|$. If S is almost normal and $q: \beta(H) \to \beta(H)/\kappa(H)$ is the quotient map, then it can be proved (Lemma 2, p. 292 of [2]) that the spectral radius of $q(S)$ is $\lim_n \sup |\alpha_n|$. Using these calculations, Arveson has proved (Corollary, p. 292 of [2]) that if

$$\limsup_n |\alpha_n| < \sup_n |\alpha_n|, \tag{i}$$

and $\mathscr{S} = \{I, S\}$, then $\mathrm{id}|_{\mathscr{S}}$ has a unique completely positive linear extension to $C^*(\mathscr{S})$, while if

$$\limsup_n |\alpha_n| = \sup_n |\alpha_n|, \tag{ii}$$

and $\mathscr{S} = \{I, S, S^2, ...\}$, then $\mathrm{id}|_{\mathscr{S}}$ does not have a unique completely positive linear extension to $C^*(\mathscr{S})$. Hence our Theorem 3.2 shows that in case (i), $\{I, S\}$ is a weak Korovkin set in $\beta(H)$, while in case (ii), even the larger set $\{I, S, S^2, ...\}$ is not a weak Korovkin set in $\beta(H)$. However, by Corollary 3.7, we see that $\{I, S, S^*S + SS^*\}$ is a weak Korovkin set in $\beta(H)$. To cite concrete cases, let $H = l^2$ and

$$S_1(x_0, x_1, ...) = (0, x_0, x_1, ...)$$

and

$$S_2(x_0, x_1, ...) = (0, 2x_0, \tfrac{3}{2}x_1, \tfrac{4}{3}x_2, ...).$$

Then $\{I, S_1, S_1^*S_1\}$ is a weak Korovkin set in $\beta(H)$, but $\{I, S_1\}$ is not. On the other hand, $\{I, S_2\}$ is a weak Korovkin set in $\beta(H)$. These results give exact noncommutative analogues of the classical theorem of Korovkin regarding approximation of positive linear operators on 1, x, and x^2.

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